Lecture 21

Wednesday, October 26, 2016 10:49 AM

Sigma Notation

- Convenient way of expressing sums using Σ called sigma notation.

DEFN If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m, n are integers such that $m \leq n$ then:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

$\sum_{i=m}^{n} \equiv$ sum, $i \equiv$ index takes on consecutive integer values betn m and n

Ex

1) $\sum_{i=4}^{7} 2i = 2.4 + 2.5 + 2.6 + 2.7 = 44$

2) $\sum_{j=7}^{11} 3 = 3 + 3 + 3 + 3 + 3 = 15$

3) $\sum_{K=1}^{4} \frac{K-1}{K^2} = \frac{1-1}{1^2} + \frac{2-1}{2^2} + \frac{3-1}{3^2} + \frac{4-1}{4^2} = \frac{1}{4} + \frac{2}{9} + \frac{3}{16} = \frac{95}{144}$

4) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$

Ex Write $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{S}$ in sigma notation.

$\sum_{i=1}^{S} \frac{1}{S}$ or $\sum_{J=0}^{S-1} \frac{1}{J+1}$

Wednesday, October 26, 2016 11:00 AM

Rules for sigma notation

Thm c is a constant that does not depend on i.

Then,

a) $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$

b) $\sum_{i=m}^{n} a_i \pm b_i = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$

Pf a)

$$\sum_{i=m}^{n} ca_i = ca_m + ca_{m+1} + \dots + ca_n$$

$$= c\left[a_m + a_{m+1} + \dots + a_n\right]$$

$$= c\sum_{i=m}^{n} a_i$$

Ex $\sum_{i=1}^{n} 1 = \underbrace{1 + \dots + 1}_{n \text{ terms}} = n$

Ex Show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$S = \sum_{i=1}^{n} i$$

$$S = 1 + 2 + 3 + \dots + n$$

$$S = n + n-1 + n-2 + \dots + 1$$

$$2S = \overbrace{(n+1) + (n+1) + \dots + (n+1)}^{n\text{-times}}$$

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

Wednesday, October 26, 2016 11:35 AM

Ex $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Soln Let $S = \sum_{i=1}^{n} i^2$

Trick $\sum_{i=1}^{n} \left[(1+i)^3 - i^3\right]$

$$= 2^3 - 1^3 + 3^3 - 2^3 + 4^3 - 3^3 + \ldots + (n+1)^3 - n^3$$

$$= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

$$\sum_{i=1}^{n} (1+i)^3 - i^3 = \sum_{i=1}^{n} 1 + 3i + 3i^2$$

$$= \sum_{i=1}^{n} 1 + 3\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} i^2$$

$$= n + \frac{3n(n+1)}{2} + 3S$$

$$= n + \frac{3n^2}{2} + \frac{3n}{2} + 3S = \frac{5n}{2} + 3n^2 + 3S$$

So,

$$n^3 + 3n^2 + 3n = 3S + \frac{3}{2}n^2 + \frac{5}{2}n$$

$$\Rightarrow 3S = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$$

$$\Rightarrow S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Rmk Principal of mathematical induction.

Ex $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Use $\sum_{i=1}^{n} (i+1)^4 - i^4$

Wednesday, October 26, 2016 11:45 AM

Ex $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(1+\frac{12i^2}{n^2}\right)\right]$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} + \frac{24i^2}{n^3}$

$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{24}{n^3}\sum_{i=1}^{n} i^2$

$= \lim_{n\to\infty} \frac{2}{n}\cdot n + \frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$= \lim_{n\to\infty} \frac{2n}{n} + \frac{24}{6}\,\frac{n}{n}\cdot\frac{(2n+1)}{n}\cdot\frac{(n+1)}{n}$

$= 2 + 4.1.2.1 = 2+8 = 10.$

Ex $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}.\,3\left[2+\frac{i}{n}\right]^2 = 23$

Ex $\sum_{i=1}^{n} i(i+1)(i+2)$